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A SHARPER TITS ALTERNATIVE FOR 3-MANIFOLD GROUPS

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ABSTRACT

The following theorem is proven. Let M be a closed, orientable, irreducible 3-manifold such that rank $H_1(M, \mathbb{Z}/p\mathbb{Z}) \geq 3$ for some prime p. Then either $\pi_1(M)$ is virtually solvable or it contains a free group of rank 2.

Introduction

The purpose of this paper is to sharpen results of Shalen–Wagreich [9] and Turaev [11]. In Theorem 2.9 of [9] Shalen and Wagreich prove that if M is a closed, orientable, irreducible 3-manifold such that $\text{rank}H_1(M, \mathbb{Z}/p\mathbb{Z}) \geq 4$ for some prime p, then $\pi_1(M)$ contains a free group of rank 2. The main result of the present paper is that if M is a closed, orientable, irreducible 3-manifold such that $\text{rank}H_1(M, \mathbb{Z}/p\mathbb{Z}) \geq 3$ for some prime p, then either $\pi_1(M)$ is virtually solvable or it contains a free group of rank 2. Combining this with results of Milnor [8] and Wolf [12], it follows that if M is a closed, orientable, irreducible 3-manifold such that $\text{rank}H_1(M, \mathbb{Z}/p\mathbb{Z}) \geq 3$ for some prime p, then either $\pi_1(M)$ is virtually nilpotent or it has exponential growth. This sharpens Shalen and Wagreich's Proposition 4.1. It also sharpens Turaev's Remark 1.IV in [11]. There it is stated that if $\text{rank}H_1(M, \mathbb{Z}/p\mathbb{Z}) \geq 3$ for some prime p, then either $\pi_1(M)$ is virtually nilpotent or for a finite set of generators of $\pi_1(M)$ there is a real number $c > 1$ such that for n large enough the number of elements in $\pi_1(M)$ of length at most n exceeds $c^{n/\log(n)}$.

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The new ingredient in the present proof is the use of p-adic analytic groups. Here is an indication of how this is done. Let $\Gamma = \pi_1(M)$, and let p be a prime as above. Let $\Gamma_1 = \Gamma$, and for $n \geq 1$ let

$$\Gamma_{n+1} = \langle (x,y)z^p : x \in \Gamma, y, z \in \Gamma_n \rangle,$$

where $(x,y) = x^{-1}y^{-1}xy$. It might be said that the Γ_n's form the p-adic lower central series of Γ. The indices on the Γ_n's are chosen to agree with the notation of Lazard's [5], which, unfortunately, does not agree with the notation of Shalen and Wagreich's [9], where $\Gamma = \Gamma_0$. The argument proceeds to the point where it may be assumed that Γ contains neither a free Abelian group of rank 2 nor a free group of rank 2. Results of Shalen and Wagreich lead to the further assumption that $\mathrm{rank}(\Gamma_n/\Gamma_{n+1}) = 3$ for every n. After possibly replacing Γ by Γ_2, Lazard's [5] shows that the completion $\hat{\Gamma}$ of Γ with respect to the Γ_n's is a p-adic analytic group of rank 3. A result in Baumslag and Shalen's [1] is used to show that Γ embeds in $\hat{\Gamma}$. Thus after it is seen that the center of Γ is trivial, the adjoint representation of Γ on the Lie algebra of $\hat{\Gamma}$ gives a faithful finite-dimensional (in fact 3-dimensional) representation of Γ over a field of characteristic 0. The Tits alternative is then used to complete the proof.

The first draft of this paper was written in ignorance of the paper [7] of Mess. The main result of this paper follows from Propositions 1 and 3 of [7]. This paper and [7] cover much the same ground, but [7] covers more. For example, under the hypotheses of Theorem 1.1 below Proposition 3 of [7] states that either $\pi_1(M)$ is virtually solvable or there exists a prime p such that M has finite covers M' with $\mathrm{rank} H_1(M', \mathbb{Z}/p\mathbb{Z})$ arbitrarily large. The proofs in both papers use p-adic analytic groups. Mess uses results of Lubotzky in [6], while the present paper deals directly with Lazard's [5].

Since the first draft of this paper was written the book [2] by Dixon, du Sautoy, Mann and Segal appeared. It contains a fine exposition of the theory of p-adic analytic groups, and it can be used in place of Lazard's [5] for the purposes of this paper. One such way to apply [2] is as follows. Lines (1.2) and (1.4) easily show that $\{\Gamma_n : n = 1, 2, 3, \ldots\}$ is a p-congruence system as in Definition 6.1 of [2] for the group Γ in (1.3). The argument preceding line (1.3) can be extended to prove that the above p-congruence system is uniformly finitely generated as in Definition 6.2 of [2]. Theorem 6.3 of [2] now implies that Γ has a faithful p-adic linear representation.

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1. The theorem and proof

THEOREM 1.1: *Let M be a closed, orientable, irreducible 3-manifold such that* $\text{rank}H_1(M, \mathbb{Z}/p\mathbb{Z}) \geq 3$ *for some prime p. Then either $\pi_1(M)$ is virtually solvable or it contains a free group of rank 2.*

Proof: Set $\Gamma = \pi_1(M)$. Just as in the proof of Theorem 2.9 of [9], the proof will separate into two cases depending on whether Γ does or does not contain a free Abelian subgroup of rank 2.

First suppose that Γ contains a free Abelian group of rank 2. By Proposition 2.8 of [9], M is either a sufficiently large manifold or a Seifert fibered space.

Suppose that M is sufficiently large. Combining the Sphere Theorem, given in Theorem 4.3 of [4], with the irreducibility of M shows that $\pi_2(M) = 0$. Thus Corollary 4.10 of [3] implies that Γ is solvable or it contains a free group of rank 2, as desired.

Suppose that M is Seifert fibered. According to Theorem 12.2 of [4], Γ contains an infinite cyclic normal subgroup Δ such that Γ/Δ is a Fuchsian group. Thus Γ/Δ is isomorphic to a subgroup of the group of all isometries of the hyperbolic plane. Since this latter group is isomorphic to the matrix group $0^1(1,2), \Gamma/\Delta$ satisfies the Tits alternative, Corollary 1 of [10], and so it easily follows that Γ does also. This concludes the proof of Theorem 1.1 if Γ contains a free Abelian group of rank 2.

Henceforth assume that Γ does not contain a free Abelian group of rank 2. Furthermore, assume that Γ does not contain a free group of rank 2. It must be shown that Γ is virtually solvable.

Recall from the introduction that $\Gamma_1 = \Gamma$ and for $n \geq 1$ that

$$\Gamma_{n+1} = \langle (x,y)z^p : x \in \Gamma, y, z \in \Gamma_n \rangle,$$

where $(x,y) = x^{-1}y^{-1}xy$. These subgroups of Γ are the same as those which appear in [9], but the indices do not agree.

Theorem 2.9 of [9] implies that $\text{rank}(\Gamma/\Gamma_2) = 3$. Lemma 1.3 of [9] implies that $\text{rank}(\Gamma_2/\Gamma_3) \geq 3$. As in the first sentence of this paragraph, Theorem 2.9 of [9] easily implies that $\text{rank}(\Gamma_2/\Gamma_3) = 3$ and $\Gamma_3 = (\Gamma_2)_2$.

The following statement will next be proven by induction on n.

$$(1.2) \qquad \operatorname{rank}(\Gamma_n/\Gamma_{n+1}) = 3 \quad \text{and} \quad \Gamma_{n+1} = (\Gamma_n)_2 \quad \text{for } n \geq 1.$$

Since the proof of (1.2) has just been completed for $n = 1$ or 2, assume that $n > 2$ and that (1.2) is true for $n-1$: $\operatorname{rank}(\Gamma_{n-1}/\Gamma_n) = 3$ and $\Gamma_n = (\Gamma_{n-1})_2$. Observe that Γ_{n-1} satisfies all the above assumptions satisfied by Γ. Hence $\operatorname{rank}((\Gamma_{n-1})_2/(\Gamma_{n-1})_3) = 3$ and $(\Gamma_{n-1})_3 = (\Gamma_n)_2$. Thus to prove (1.2), it suffices to prove that $\Gamma_{n+1} = (\Gamma_n)_2$.

It is clear that $(\Gamma_n)_2 \subseteq \Gamma_{n+1}$, so to prove that $\Gamma_{n+1} = (\Gamma_n)_2$, it suffices to prove that $\Gamma_{n+1} \subseteq (\Gamma_n)_2$. In turn it suffices to show that $(x, v) \in (\Gamma_n)_2$ for all elements $x \in \Gamma$ and $v \in \Gamma_n$. Since $\Gamma_n = (\Gamma_{n-1})_2$ by induction, it suffices to prove that $(x, (y, w)) \in (\Gamma_n)_2$ and $(x, y^p) \in (\Gamma_n)_2$ for all elements $x \in \Gamma$ and $y, w \in \Gamma_{n-1}$.

First consider the assertion $(x, (y, w)) \in (\Gamma_n)_2$, where $x \in \Gamma$ and $y, w \in \Gamma_{n-1}$. Set $z = w^y = y^{-1}wy$. Line (II.1.1.6.3) of [5] states that

$$(x^y, (y, z))(y^z, (z, x))(z^x, (x, y)) = 1.$$

Since $z^x \in \Gamma_{n-1}$ and $(x, y) \in \Gamma_n = (\Gamma_{n-1})_2$, $(z^x, (x, y)) \in (\Gamma_{n-1})_3 = (\Gamma_n)_2$. Likewise, $(y^z, (z, x)) \in (\Gamma_n)_2$. Thus $(x^y, (y, z)) \in (\Gamma_n)_2$. Thus $(\Gamma_n)_2$ contains $(x^y, (y, z))^{y^{-1}} = (x, (y, z^{y^{-1}})) = (x, (y, w))$, as desired.

Second consider the assertion $(x, y^p) \in (\Gamma_n)_2$, where $x \in \Gamma$ and $y \in \Gamma_{n-1}$. Line (II.1.1.6.2) of [5] states that

$$(x, yz) = (x, z)(x, y)^z.$$

Hence

$$(x, y^p) = (x, y)(x, y^{p-1})^y.$$

Since $(x, y^{p-1}) \in \Gamma_n = (\Gamma_{n-1})_2$, it follows that

$$(x, y^{p-1})^y \equiv (x, y^{p-1}) \mod (\Gamma_{n-1})_3.$$

Since $(\Gamma_{n-1})_3 = (\Gamma_n)_2$,

$$(x, y^p) \equiv (x, y)(x, y^{p-1}) \mod (\Gamma_n)_2.$$

Continuing in this way,

$$(x, y^p) \equiv (x, y)^p \equiv 1 \mod (\Gamma_n)_2.$$

This completes the proof of (1.2).

It is well-known and can be proven using commutator identities as above that $(\Gamma_m, \Gamma_n) \subseteq \Gamma_{m+n}$ for all positive integers m, n. Thus the sequence of subgroups $\Gamma_1, \Gamma_2, \Gamma_3, \ldots$ gives Γ the structure of a filtered group, as defined in (II.1.1) of [5]. Let ω denote the filtration function of Γ as in [5].

Now let n be an integer with $n \geq 2$. Since $(\Gamma_n, \Gamma_n) \subseteq \Gamma_{2n}$, Γ_n/Γ_{n+2} is an Abelian group. Since $\operatorname{rank}(\Gamma_n/\Gamma_{n+1}) = 3$, $\operatorname{rank}(\Gamma_{n+1}/\Gamma_{n+2}) = 3$ and $\Gamma_{n+1} = (\Gamma_n)_2$, it easily follows that $\Gamma_n/\Gamma_{n+2} \cong (\mathbb{Z}/p^2\mathbb{Z})^3$. Thus if x is an element of Γ with $\omega(x) = n$, then the image of x in Γ_n/Γ_{n+2} has order p^2. Hence $\omega(x^p) = n+1$. Because Γ_2 satisfies all the above assumptions satisfied by Γ, the discussion in this paragraph gives the following by replacing Γ by Γ_2 if necessary:

$$(1.3) \qquad \omega(x) > (p-1)^{-1} \text{ and } \omega(x^p) = \omega(x) + 1 \quad \text{for every } x \in \Gamma.$$

Replacing Γ by Γ_2 causes another small difficulty in notation. If such a replacement is made, the filtration does not change — it is the filtration induced from the original group. The notation of (II.1.1) of [5] will be maintained regarding the subgroups Γ_n. Thus although the Γ_n's still form the p-adic lower central series of Γ, their indices are shifted by 1.

In this paragraph it will be shown that $\Gamma_\infty = 1$, namely,

$$(1.4) \qquad \omega(x) < \infty \quad \text{for every nontrivial element } x \text{ in } \Gamma.$$

Corollary A1 of [1] will be used to prove this. It shows that since Γ is the fundamental group of an irreducible, orientable 3-manifold and Γ does not contain a free Abelian group of rank 2, every infinite-index subgroup of Γ generated by at most 2 elements is free (of rank at most 2). Since Γ does not contain a free group of rank 2, this free group must in fact have rank at most 1. Now let x be an element of Γ with $\omega(x) = \infty$ and let y be an element of Γ with $\omega(y) < \infty$. The subgroup $\langle x, y\rangle$ of Γ generated by x and y has infinite index in Γ because its image in Γ/Γ_∞ is cyclic and $\operatorname{rank}(\Gamma_n/\Gamma_{n+1}) = 3$ for $n \geq 2$. Thus $\langle x, y\rangle$ is infinite cyclic. However, the second assertion in (1.3) shows that the image of $\langle x, y\rangle$ in Γ/Γ_∞ is also infinite cyclic, and so the kernel of the canonical homomorphism from $\langle x, y\rangle$ to Γ/Γ_∞ must be trivial, namely, $x = 1$. This proves (1.4).

Observe that (1.3) and (1.4) imply that Γ is torsion-free. It easily follows that

$$(1.5) \qquad \text{the center of } \Gamma \text{ is trivial}$$

because any nontrivial element in the center of Γ and any element in Γ not in the subgroup generated by the first element generate a subgroup isomorphic with $\mathbb{Z}^2$, which does not exist.

By Definition (III.2.1.2) of [5], lines (1.3) and (1.4) show that Γ is a p-valued group. Furthermore, it has rank 3 because $\mathrm{rank}(\Gamma_n/\Gamma_{n+1}) = 3$ for $n \geq 2$. Thus it is easy to see that the completion $\hat{\Gamma}$ of Γ with respect to the Γ_n's is also p-valued of rank 3. Line (1.4) implies that Γ embeds in $\hat{\Gamma}$. Proposition (III.2.1.8) of [5] shows that $\hat{\Gamma}$ is p-saturated. Theorem (III.3.3.2) of [5] now shows that $\hat{\Gamma}$ is a p-adic analytic group of rank 3. Section (IV.3.2) of [5] now associates to $\hat{\Gamma}$ a 3-dimensional Lie algebra. The adjoint representation of $\hat{\Gamma}$ on its Lie algebra obtains a 3-dimensional representation of Γ over a field of characteristic 0. Line (1.5) easily shows that this representation is faithful. According to the Tits alternative, either Γ contains a free group of rank 2 or it is virtually solvable. This completes the proof of Theorem 1.1. ∎

COROLLARY 1.6: *Let M be a closed, orientable, irreducible 3-manifold such that* $\mathrm{rank} H_1(M, \mathbb{Z}/p\mathbb{Z}) \geq 3$ *for some prime p. Then either $\pi_1(M)$ is virtually nilpotent or it has exponential growth.*

Proof: This is an immediate consequence of Theorem 1.1 and Theorem 4.3 of [12] and the main theorem of [8]. ∎

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